University of Bahrain

SERIAL

College of Information Technology
Department of Computer Science

ITCS253 Discrete Structures II

First Semester 2015/2016

Second Exam – 75 Minutes

*** Key Solution ***

STUDENT NAME	**** **Key Solution** ****
STUDENT#	**** **Key Solution** ****
SECTION	**** **Key Solution** ****

- ▶ This exam contains **5 pages** (including this cover page) and **5 questions**. Check to see if any pages are missing.
- ▶ You are **allowed** to use Calculators.
- ▶ You are **not allowed** to use books, notes, or mobiles.
- ▶ Please write **one answer**. In case of writing multiple answers, mistakes in any answer will be counted.

Question	Points	Score
1	5	
2	4	
3	5	
4	8	
5	8	
Total:	30	

Instructor: Dr. Ali Alsaffar Sections# 1 & 2

Answer all questions

(1) Answer the following questions.

(a) **[1 point]** Give an example of a recurrence relation of order 3 with constant coefficients but neither homogeneous nor linear.

Solution: $a_n = 2\sqrt{a_{n-1}} + 3a_{n-2} - 4a_{n-3} + 1$

(b) **[1 point]** Find the *characteristic equation* of $a_n = 6a_{n-1} + 3^{n+1}$.

Solution: $(x-6)(x-3) = 0$

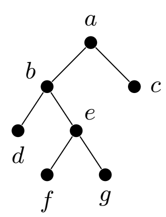
(c) **[1 point]** Suppose a recurrence relation $a_n = 2(3)^n + 4(7)^n$. Write a_n as a linear recurrence relation with constant coefficients.

Solution: The characteristic equation is $(x-3)(x-7) = x^2 - 10x + 21 = 0$.
Possible recurrence relations are:

1. $a_n - 10a_{n-1} + 21a_{n-2} = 0$.
2. $a_n - 3a_{n-1} = c \cdot 7^n$, where c is a constant.
3. $a_n - 7a_{n-1} = c \cdot 3^n$, where c is a constant.

(d) **[1 point]** Draw a binary tree that is full but not balanced.

Solution:



(e) **[1 point]** Write the degree equation for a directed graph.

Solution: Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

(2) **[4 points]** Suppose a graph G has n vertices, two vertices of degree one and the rest each has a loop and adjacent to exactly 3 different vertices (no parallel edges). (a) If the number of edges is 3746 find the number of vertices n. (b) Is the graph a tree? Why?

Solution: (a) $\sum_{v \in V} deg(v) = 2 + 5 \times (n-2) = 2e \implies n = (2e+8)/5 = (2 \times 3746 + 8)/5 = 1500$.
(b) No. Because $e \neq n - 1$.

(3) Let T be a full 4-*arry* tree with n vertices, e edges and t terminals (leaves.)

(a) **[3 points]** Show that $t = \dfrac{3n+1}{4}$.

(b) **[2 points]** Suppose T is of height h. Show that $e \leq (\frac{4}{3})(4^h - 1)$.

Solution:

1. For a full m-*arry* tree, we know that $n = mi + 1$ and $n = i + t$, where i is the number of internal vertices. Hence, $i = n - t$, substitute in $n = mi + 1$
   $\implies n = m(n-t) + 1$
   $\implies mt = n(m-1) + 1$
   $\implies t = \dfrac{n(m-1)+1}{m}$
   Since $m = 4$, then $t = \dfrac{n(4-1)+1}{4} = \dfrac{3n+1}{4}$.

2. For a full $m - arry$ tree with height h, the maximum number of vertices is $\dfrac{m^{h+1}-1}{m-1}$.
   Since $m = 4$, then $n_{max} = \dfrac{4^{h+1}-1}{3}$. Hence, for a tree with n vertices $n \leq n_{max}$. But $e = n - 1$ and $n = e + 1$, then $e + 1 \leq n_{max}$
   $\implies e + 1 \leq \dfrac{4^{h+1}-1}{3}$
   $\implies e \leq \dfrac{4^{h+1}-1}{3} - 1$
   $\implies e \leq \dfrac{4^{h+1}-1-3}{3} = \dfrac{4(4^{h+1}-1)}{3}$
   $\implies e \leq (\frac{4}{3})(4^h - 1)$.

(4) (a) **[5 points]** Use the characteristic equation to solve the following recurrence relation.

$$t_0 = 0, \quad t_n = 3t_{n-1} + 2, \quad n \geq 1$$

Solution:

By re-writing the relation: $t_n - 3t_{n-1} = 2$

The characteristic equation is $(x-3)(x-1) = 0$

$\therefore t_n = c_1(3)^n + c_2$

$t_1 = 3t_0 + 2 = 2$. Thus,

$$\begin{aligned} t_0 = 0 \rightarrow &\quad c_1 + c_2 = 0 \\ t_1 = 2 \rightarrow &\quad 3c_1 + c_2 = 2. \end{aligned}$$

Subtract the two equations $2c_1 = 2 \implies c_1 = 1$ and $c_2 = -1$.

As a result, $t_n = 3^n - 1$.

(b) **[3 points]** Use the characteristic equation to solve the following recurrence relation without finding the constants.

$$a_n = 5a_{n-1} + 3^n(5^n - 1)$$

Solution:

$a_n - 5a_{n-1} = 3^n \cdot 5^n - 3^n = (15)^n - 3^n$.

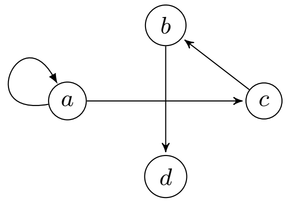
The characteristic equation is $(x-5)(x-15)(x-3) = 0$.

Then, $a_n = c_1(5)^n + c_2(15)^n + c_3(3)^n$.

(5) Suppose M is the adjacency matrix for a *directed* graph G.

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(a) **[2 points]** Draw the graph using the below vertex ordering.



(b) **[2 points]** Find the adjacency list of the graph.

Solution:

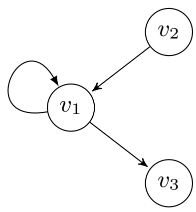
Vertex	Adjacency Vertices
a	a, c
b	d
c	b

(c) **[1 point]** What is the degree of the vertex a ?

Solution: $deg^+(a) = deg^-(a) = 1$.

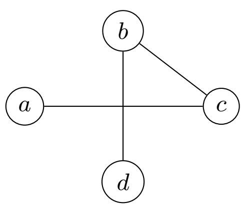
(d) **[2 points]** Is $H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ a subgraph of G ? Justify your answer.

Solution:



H is not a subgraph of G. Because v_1 can be matched only with a since they have loops. However, v_1 in H has an in-arrow but a in G has no in-arrows.

(e) **[1 point]** Assume the graph is undirected. Find a spanning tree from G.



Solution: